Relation liant la pression et le champ de vitesse dans les équations de Navier Stokes incompressibles

En prenant la divergence des équations de Navier-Stokes incompressibles :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}} \boldsymbol{\Delta} \mathbf{u}, \tag{1}$$

et en utilisant la condition d'incompressibilité :

$$\mathrm{div}\ \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2}$$

on obtient l'équation de Poisson suivante permettant d'exprimer la pression en fonction du champ de vitesse :

$$\boxed{\boldsymbol{\Delta} p = \sum_{\substack{i=1\\j=1\\i\neq j}}^{n} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} - \sum_{\substack{i=1\\j=1\\i\neq j}}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}} \tag{3}$$

En 2D cette relation est donc donnée par :

$$\boldsymbol{\Delta} p = 2 \left(\frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right) \tag{4}$$

Et en 3D par :

$$\boldsymbol{\Delta} p = 2 \left(\left(\frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_1} \frac{\partial u_3}{\partial x_3} + \frac{\partial u_2}{\partial x_2} \frac{\partial u_3}{\partial x_3} \right) - \left(\frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3} \frac{\partial u_3}{\partial x_2} \right) \right) \tag{5}$$

Démonstration pour $n = 3$:

Calculons la divergence de chacun des termes de l'équation 1.

- div $(\partial_t \mathbf{u})$

D'après la condition d'incompressibilité 2, on a :

$$\mathrm{div} \left(\frac{\partial \mathbf{u}}{\partial t} \right) = \frac{\partial}{\partial t} (\mathrm{div}\ \mathbf{u}) = 0 \tag{6}$$

- div $(\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u})$

$$
\begin{aligned}
\operatorname{div}\,(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) &= \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \cdot \begin{pmatrix} u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} \\ u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3} \\ u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3} \end{pmatrix} \\
&= \left(\color{blue}{\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}} + \color{red}{u_1\frac{\partial^2 u_1}{\partial x_1^2}}\right) + \left(\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \color{orange}{u_2\frac{\partial^2 u_1}{\partial x_1 \partial x_2}}\right) + \left(\frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \color{purple}{u_3\frac{\partial^2 u_1}{\partial x_1 \partial x_3}}\right) \\
&+ \left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \color{red}{u_1\frac{\partial^2 u_2}{\partial x_2 \partial x_1}}\right) + \left(\color{blue}{\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2}} + \color{orange}{u_2\frac{\partial^2 u_2}{\partial x_2^2}}\right) + \left(\frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \color{purple}{u_3\frac{\partial^2 u_2}{\partial x_2 \partial x_3}}\right) \\
&+ \left(\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} + \color{red}{u_1\frac{\partial^2 u_3}{\partial x_3 \partial x_1}}\right) + \left(\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} + \color{orange}{u_2\frac{\partial^2 u_3}{\partial x_3 \partial x_2}}\right) + \left(\color{blue}{\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}} + \color{purple}{u_3\frac{\partial^2 u_3}{\partial x_3^2}}\right) \\
&= \color{blue}{\alpha} + \beta + \color{red}{\gamma} + \color{orange}{\delta} + \color{purple}{\epsilon}
\end{aligned}
$$

Avec :

$$
\begin{aligned}
\color{blue}{\alpha} &= \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3} \\
\beta &= 2\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + 2\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} + 2\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} \\
\color{red}{\gamma} &= u_1\frac{\partial^2 u_1}{\partial x_1^2} + u_1\frac{\partial^2 u_2}{\partial x_2 \partial x_1} + u_1\frac{\partial^2 u_3}{\partial x_3 \partial x_1} \\
\color{orange}{\delta} &= u_2\frac{\partial^2 u_2}{\partial x_2^2} + u_2\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + u_2\frac{\partial^2 u_3}{\partial x_3 \partial x_2} \\
\color{purple}{\epsilon} &= u_3\frac{\partial^2 u_3}{\partial x_3^2} + u_3\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + u_3\frac{\partial^2 u_2}{\partial x_2 \partial x_3}
\end{aligned}
$$

Or :

$$
\begin{aligned}
\alpha &= \frac{\partial u_1}{\partial x_1}\cancel{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_3} \\
&+ \frac{\partial u_2}{\partial x_2}\cancel{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3} \\
&+ \frac{\partial u_3}{\partial x_3}\cancel{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_2} \\
&= -2\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - 2\frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_3} - 2\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3} \\
u_1\frac{\partial^2 u_1}{\partial x_1^2} &= u_1\left(\nabla x \cdot \cancel{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}\right) - u_1\frac{\partial^2 u_2}{\partial x_1 \partial x_2} - u_1\frac{\partial^2 u_3}{\partial x_1 \partial x_3} \\
&= -u_1\frac{\partial^2 u_2}{\partial x_1 \partial x_2} - u_1\frac{\partial^2 u_3}{\partial x_1 \partial x_3} \\
u_2\frac{\partial^2 u_2}{\partial x_2^2} &= u_2\left(\nabla y \cdot \cancel{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}\right) - u_2\frac{\partial^2 u_1}{\partial x_2 \partial x_1} - u_2\frac{\partial^2 u_3}{\partial x_2 \partial x_3} \\
&= -u_2\frac{\partial^2 u_1}{\partial x_2 \partial x_1} - u_2\frac{\partial^2 u_3}{\partial x_2 \partial x_3} \\
u_3\frac{\partial^2 u_3}{\partial x_3^2} &= u_3\left(\nabla z \cdot \cancel{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}\right) - u_3\frac{\partial^2 u_1}{\partial x_3 \partial x_1} - u_3\frac{\partial^2 u_2}{\partial x_3 \partial x_2} \\
&= u_3\frac{\partial^2 u_1}{\partial x_3 \partial x_1} - u_3\frac{\partial^2 u_2}{\partial x_3 \partial x_2}
\end{aligned}
$$

Les matrices jacobiennes et hessiennes de $\mathbf{u}$ sont des matrices symétriques, donc $\gamma = \delta = \epsilon = 0$.

Ainsi :

$$
\begin{aligned}
\text{div }(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) &= \alpha+\beta \\
&= \left(-2\frac{\partial u_1}{\partial x_1}\ \frac{\partial u_2}{\partial x_2} - 2\frac{\partial u_1}{\partial x_1}\ \frac{\partial u_3}{\partial x_3} - 2\frac{\partial u_2}{\partial x_2}\ \frac{\partial u_3}{\partial x_3}\right) + \left(2\frac{\partial u_2}{\partial x_1}\ \frac{\partial u_1}{\partial x_2} + 2\frac{\partial u_1}{\partial x_3}\ \frac{\partial u_3}{\partial x_1} + 2\frac{\partial u_2}{\partial x_3}\ \frac{\partial u_3}{\partial x_2}\right)
\end{aligned}
$$

- div (-$\boldsymbol{\nabla}p$) = -$\boldsymbol{\Delta}p$

- div $\left(\frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\mathbf{u}\right) = \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}(\text{div }\mathbf{u}) = 0$

Par conséquent :

$$
\begin{aligned}
&\text{div }\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}p + \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\mathbf{u}\right) \\
&\iff \text{div }(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) = \text{div }(-\boldsymbol{\nabla}p) \\
&\iff \boldsymbol{\Delta}p = \left(2\frac{\partial u_1}{\partial x_1}\ \frac{\partial u_2}{\partial x_2} + 2\frac{\partial u_1}{\partial x_1}\ \frac{\partial u_3}{\partial x_3} + 2\frac{\partial u_2}{\partial x_2}\ \frac{\partial u_3}{\partial x_3}\right) - \left(2\frac{\partial u_2}{\partial x_1}\ \frac{\partial u_1}{\partial x_2} + 2\frac{\partial u_1}{\partial x_3}\ \frac{\partial u_3}{\partial x_1} + 2\frac{\partial u_2}{\partial x_3}\ \frac{\partial u_3}{\partial x_2}\right) = 5
\end{aligned}
$$